Step 1. Analytic Properties of the Riemann zeta function

Step 1 Part 1; $\mathrm{Re}\, s > 1$ **[2 lectures]**

Recall that the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{6}$$

which converges (absolutely) for $\mathrm{Re}\, s > 1$. Each individual term $n^{-s} = e^{-s \log n}$ is holomorphic in $\mathbb{C}$. What about when we add infinitely many such holomorphic functions? To say the series is holomorphic requires more than it simply converges (even absolutely).

Definition 6.1 *Let $\{F_n(s)\}_{n\geq 1}$ be a sequence of functions defined on $\mathcal{S} \subseteq \mathbb{C}$. The sequence $\{F_n\}_{n\geq 1}$ **converges** to Fon $\mathcal{S}$ iff*

$$\forall \varepsilon > 0, \forall s \in \mathcal{S}, \exists N = N(\varepsilon, s) : \forall n \geq N, |F_n(s) - F(s)| < \varepsilon.$$

*The sequence $\{F_n\}_{n\geq 1}$ converges **uniformly** to Fon $\mathcal{S}$ iff*

$$\forall \varepsilon > 0, \exists N = N(\varepsilon, \mathcal{S}) : \forall s \in \mathcal{S}, \forall n \geq N, |F_n(s) - F(z)| < \varepsilon.$$

Make sure you understand the difference between these two definitions. Importantly, the N found in the second one works for all $s \in \mathcal{S}$ *simultaneously*, i.e. *uniformly*.

In applications to the Riemann zeta function the $F_n(s)$ of this definition will be the partial sums of the series defining $\zeta(s)$. To check that a series converges uniformly we can use Weierstrass's M-test:

Theorem 6.2 Weierstrass M-Test. *If $\forall n \geq 1$ there exists $M_n > 0$ such that $|f_n(s)| < M_n$ for all $s \in \mathcal{S}$, and $\sum_{n=1}^{\infty} M_n < \infty$ then $\sum_{n=1}^{\infty} g_n(s)$ converges uniformly on $\mathcal{S}$.*

Proof See the Background: Complex Analysis II notes. ■

Example 6.3 *$\zeta(s)$ converges uniformly in $\operatorname{Re} s \geq 1+\delta$ for any $\delta > 0$.*

Proof Let $\delta > 0$ be given and set $M_n = 1/n^{1+\delta}$. Then for $\operatorname{Re} s \geq 1+\delta$ we have

$$\left|\frac{1}{n^s}\right| = \frac{1}{n^\sigma} \leq \frac{1}{n^{1+\delta}}.$$

Thus

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} = \zeta(1+\delta) < 1 + \frac{1}{\delta},$$

by a result from Chapter 1. Therefore $\sum_{n=1}^{\infty} M_n$ converges and hence $\zeta(s)$ converges uniformly in $\operatorname{Re} s \geq 1+\delta$. ■

Example 6.4 *The series*

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

converges uniformly in $\operatorname{Re} s \geq 1+\delta$ for any $\delta > 0$.

Proof left to student. Apply the M-test with $M_n = (\log n)/n^{1+\delta}$. ■

What follows from *uniform* convergence? We have Weierstrass's Theorem for Series (see Background: Complex Analysis II notes, or p. 94-95 of Sansone & Gerretsen, *Lectures on the Theory of Functions of a Complex Variable, I*, Noordhoff - Groningen, 1960.) which states

Theorem 6.5 Weierstrass's Theorem for Series. *Assume the functions in the sequence $\{f_i(s)\}_{i\geq 1}$ are holomorphic in an open set $\mathcal{U} \subseteq \mathbb{C}$, and $\sum_{i=1}^{\infty} f_i(s)$ converge uniformly on every closed and bounded subset of $\mathcal{U}$. Then*

(i) $F(s) = \sum_{i=1}^{\infty} f_i(s)$ is holomorphic on $\mathcal{U}$,

(ii) for all $k \geq 1$, the series $\sum_{i=1}^{\infty} f_i^{(k)}(s)$ converges on $\mathcal{U}$, and converge uniformly on every closed and bounded subset of $\mathcal{U}$ with limit $F^{(k)}(s)$. (That is, the series can be differentiated term-by-term.)

Proof not given. ■

Applied to $\zeta(s)$ this gives

Theorem 6.6 *The Riemann zeta function is holomorphic in $Rs > 1$, with derivative*

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s}$$

for $\operatorname{Re} s > 1$. Further $\zeta'(s)$ is holomorphic in $\operatorname{Re} s > 1$.

Proof A closed and bounded subset of $s : \operatorname{Re} s > 1$ lies within $s : \operatorname{Re} s \geq 1+\delta$ for some $\delta > 0$. We have shown that the series defining $\zeta(s)$ converges uniformly in $\operatorname{Re} s \geq 1+\delta$. Thus the series defining $\zeta(s)$ converges uniformly on the closed and bounded subset of $s : \operatorname{Re} s > 1$. True for all closed and bounded subsets means we can apply Weierstrass's result to say that $\zeta(s)$ is holomorphic in $\operatorname{Re} s > 1$. The derivative follows from Part ii of Weierstrass and

$$\frac{d}{ds}\left(\frac{1}{n^s}\right) = \frac{d}{ds} e^{-s\log n} = -\frac{\log n}{n^s}.$$

■

From this we can derive

Example 6.7 *The logarithmic derivative*

$$\frac{\zeta'}{\zeta}(s)$$

is holomorphic in $\operatorname{Re} s > 1$.

Proof From its interpretation of $\zeta(s)$ as a convergent infinite product we know that $\zeta(s) \neq 0$ in $\operatorname{Re} s > 1$. Thus $1/\zeta(s)$ is well-defined for such s. That $\zeta'(s)/\zeta(s)$ is holomorphic in $\operatorname{Re} s > 1$ follows from Theorem 6.6.

■

Theorem 6.5 and Example 6.4 together give

Example 6.8 *The series*

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

is holomorphic in $\operatorname{Re} s > 1$.

In Chapter 2 we showed that

$$\frac{\zeta'}{\zeta}(\sigma) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma}, \tag{7}$$

for all *real* $\sigma > 1$. Is this true for *complex* $s : \operatorname{Re} s > 1$?

From Complex Analysis we have

Theorem 6.9 *Assume that $F(z)$ and $G(z)$ are analytic in a path connected open set $\mathcal{U}$. Assume there exists a convergent sequence $\{z_i\}_{i\geq 1}$ of points of $\mathcal{U}$ with limit point z_ℓ also in $\mathcal{D}$ for which $F(z_i) = G(z_i)$ for all $i \geq 1$ and $F(z_\ell) = G(z_\ell)$. Then $F(z) = G(z)$ for all $z \in \mathcal{U}$.*

This is the last result in the Background notes: Analytic Continuation.

Theorem 6.10 *The logarithmic derivative satisfies*

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

for $\operatorname{Re} s > 1$.

Proof In Theorem 6.9 let $F(s) = \zeta'(s)/\zeta(s)$, $G(s) = -\sum_{n=1}^{\infty} \Lambda(n)\, n^{-s}$, $\mathcal{U} = \{s : \operatorname{Re} s > 1\}$ and $\{z_i\}_{i\geq 1}$ a convergent sequence of real numbers > 1 with limit > 1, (e.g. $z_i = 3 - 1/i$). The result then follows from Theorem 6.9 and (7). ■